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THERMOELASTIC PROBLEM OF CURVILINEAR CRACKS IN BONDED DISSIMILAR MATERIALS

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Abstract—The two-dimensional problem of curvilinear cracks lying along the interface between dissimilar materials under remote heat flux is considered. Based on the Hilbert formulation and a special technique of analytical continuation, closed form solutions for the stress functions in both the inclusion and the surrounding matrix have been obtained in this study. It is shown that singularities of the thermal stresses possess the same tri-log character as those obtained for isothermal problems which would not be affected by the discontinuous jumps of the thermal properties across the interface. For illustrating the use of the present approach, detailed results are given for a single circular-arc crack in bi-material plate under uniform remote heat flux. Both the stress functions and stress intensity factors are expressed in an explicit form and the latter are verified by comparison with the existing ones. Numerical examples for commonly used fiber-reinforced composites such as boron/epoxy, carbon/epoxy and glass/epoxy systems associated with an interface circular-arc crack are examined and detailed results are provided. The validity of the fully open crack assumption is also discussed.

1. INTRODUCTION

In view of the widespread use of high-temperature composite materials in advanced engineering structures, the damage tolerance and reliability of composite material and structures have become matters of concern. There arose the problem of finding thermal stress distribution in bonded dissimilar materials containing imperfections in the form of interface cracks. The steady-state thermoelastic problems of interface cracks between dissimilar isotropic media have been studied by Erdogan (1965), Barber and Comninou (1982, 1983), Martin-Moran *et al.* (1983), Sumi and Ueda (1990). As to the cracks between anisotropic media, solutions were given by Clements (1983), Hwu (1992), Ting *et al.* (1992) and Chao and Chang (1993). Although the thermoelastic problems of interface crack have been studied extensively over the past 30 years, very few published analytical studies are available for the corresponding problems associated with curvilinear cracks. Based on the properties of Plemelj formula and Cauchy integrals, closed form solutions for curvilinear cracks in bonded dissimilar materials under inplane load and bending have been given by Perlman and Sih (1967a, b). They found that the stresses near the tips of a curved crack possess the same trig-log character of singularity as those obtained for a straight crack between dissimilar materials. Recently, Chao and Shen (1993a) solved the thermoelastic problem of curvilinear cracks in isotropic medium by application of the complex variable theory dealing with sectionally holomorphic functions. It was found that the thermal stresses near the tips of a curved crack possess the same character of singularity as those for a straight crack.

In the present study, we aim to provide the general solution to the thermoelastic problem of curvilinear cracks in bonded dissimilar materials. Based upon the Hilbert problem formulation and a special technique of analytical continuation, the stress functions pertaining to each material medium are obtained in closed form. Details of the solution are given for a single circular-arc crack in a bi-material plate under remote heat flux. Both the stress functions and stress intensity factors are expressed in an explicit form. It should be noted that the thermoelastic field presented here may become invalid for certain combinations of the angle of heat flux and central angle subtended by the circular-arc crack. The presence of a closed form expression of the displacements enables us to find the critical crack angle for which the crack faces come into contact with one another. Three typical examples of composite materials are considered in the following work to illustrate the use of the present approach.

2. STATEMENT OF THE PROBLEM

Consider two homogeneous, isotropic elastic materials. Let one occupy the region S^+, interior to the unit circle, $r = 1$, while the other occupies the infinite region S^-, exterior to the unit circle (Fig. 1). The thermoelastic properties of the material in S^+ can be specified by the constants μ_1, α_1, k_1, κ_1 and those of the material S^- by μ_2, α_2, k_2, κ_2 where μ_j, α_j and k_j are the shear modulus, thermal expansion coefficient, and heat conductivity, respectively and $\kappa_j = (3-v_j)/(1+v_j)$ for generalized plane stress and $\kappa_j = (3-4v_j)$ for plane strain, v_j being the Poisson's ratio ($j = 1, 2$). If the bond between the two materials on the unit circle is imperfect, it can be represented as the sum of L and L^*, where $L = L_1+L_2+\cdots+L_n$, the union of n circular-arc cracks a_mb_m, $m = 1, 2, \ldots, n$.

and:

$$L^* = L_1^*+L_2^*+\cdots+L_n^*, \text{ the union of } n \text{ circular-arc bond } b_ma_{m+1},$$
$$m = 1,2,\ldots,n \quad \text{and} \quad a_{n+1} = a_1.$$

Let the center of the unit circle be placed at the origin of the complex plane, $z = x+iy$ and $t = \exp(i\varphi)$ be those points of z on $|z| = 1$. For this problem, the tractions σ_r, $\tau_{r\theta}$ will be specified on L while the continuity of the stresses and displacements are required on L^*, i.e.

$$(\sigma_r^+)_1+i(\tau_{r\theta}^+)_1 = p^+(t), \quad \text{on } L, \tag{1}$$

$$(\sigma_r^-)_2+i(\tau_{r\theta}^-)_2 = p^-(t), \quad \text{on } L, \tag{2}$$

and

$$(\sigma_r)_1+i(\tau_{r\theta})_1 = (\sigma_r)_2+i(\tau_{r\theta})_2, \quad \text{on } L^* \tag{3}$$

$$u_1+iv_1 = u_2+iv_2, \quad \text{on } L^*. \tag{4}$$

The superscripts + and − in eqn (1)–(2) are used to denote the boundary values of the stresses as they are approached from S^+ and S^-, respectively, and the quantities $p^+(t)$, $p^-(t)$ are the prescribed tractions on the crack surfaces. The polar components of the stress tensor and displacement vector in the two-dimensional theory of isotropic thermoelasticity can be expressed in terms of the complex functions $\Phi_j(z)$ and $\Psi_j(z)$. Extending the Bogdanoff (1954) stress combinations to the regions S^+ and S^-, it follows that:

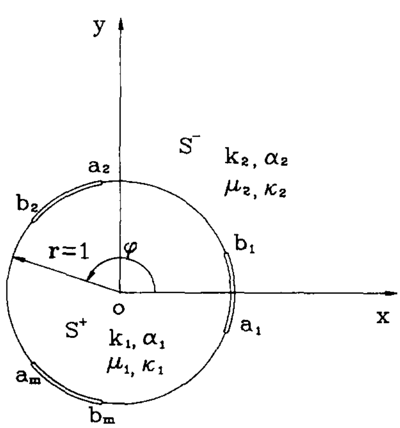

Fig. 1. An infinite medium partially bonded to a circular insert.

$$(\sigma_r)_j + (\sigma_\theta)_j = 2[\Phi_j(z) + \overline{\Phi_j(z)}], \tag{5}$$

$$(\sigma_r)_j + i(\tau_{r\theta})_j = \Phi_j(z) + \overline{\Phi_j(z)} - \overline{z\Phi_j'(z)} - \left(\frac{\bar{z}}{z}\right)\overline{\Psi_j(z)}, \tag{6}$$

and, the displacements may be combined to give:

$$2\mu_j(u_j + iv_j) = \kappa_j\phi_j(z) - z\bar{\phi}_j'(\bar{z}) - \bar{\psi}_j(\bar{z}) + 2\mu_j\beta_j g_j(z), \tag{7}$$

with

$$j = \begin{matrix} 1, & \text{for } z \in S^+ \\ 2, & \text{for } z \in S^- \end{matrix}\Bigg\}$$

$$\phi'(z) = \Phi(z), \quad \psi'(z) = \Psi(z), \quad g_j(z) = \int \phi_{0_j}(z)\,\mathrm{d}z, \quad \beta_j = (1+\nu_j)\alpha_j,$$

$$\phi_0(z) = T(x,y) + iW(x,y),$$

where the overbar denotes the conjugate of the complex function, $T(x, y)$ stands for the temperature and $W(x, y)$ the harmonic conjugate of the temperature.

For problems involving arcs of discontinuities, it is convenient to further introduce the functions:

$$\Omega_j(z) = \bar{\Phi}_j\left(\frac{1}{z}\right) - \frac{1}{z}\bar{\Phi}_j'\left(\frac{1}{z}\right) - \frac{1}{z^2}\bar{\Psi}_j\left(\frac{1}{z}\right), \quad j = 1,2. \tag{8}$$

Making use of eqns (6) and (8), (1)–(4) can now be expressed in terms of $\Phi_j(z)$, $\Omega_j(z)$ as:

$$\Phi_1^+(t) + \Omega_1^-(t) = p^+(t), \quad \text{on } L, \tag{9}$$

$$\Phi_2^-(t) + \Omega_2^+(t) = p^-(t), \quad \text{on } L, \tag{10}$$

and

$$\Phi_1(t) + \Omega_1(t) = \Phi_2(t) + \Omega_2(t), \quad \text{on } L^*, \tag{11}$$

$$\frac{1}{\mu_1}[\kappa_1\Phi_1(t) - \Omega_1(t) + 2\mu_1\beta_1 g_1'(t)] = \frac{1}{\mu_2}[\kappa_2\Phi_2(t) - \Omega_2(t) + 2\mu_2\beta_2 g_2'(t)], \quad \text{on } L^*. \tag{12}$$

It should be pointed out that eqn (12) requires only the derivative:

$$\frac{\partial u_1}{\partial \phi} + i\frac{\partial v_1}{\partial \phi} = \frac{\partial u_2}{\partial \phi} + i\frac{\partial v_2}{\partial \phi},$$

to be continuous across L^* instead of the displacements u_j, v_j themselves as indicated in eqn (4). Thus, eqn (12) will satisfy eqn (4) to within an additive constant. Hence, a complete solution to the bi-material crack problem has been reduced to the evaluation of four complex functions $\Phi_j(z)$, $\Omega_j(z)$, $(j = 1, 2)$, which must satisfy the conditions as given by eqns (9)–(12).

3. PROPERTIES OF THE COMPLEX FUNCTIONS

A knowledge of the behavior of the complex functions for small and large values of $|z|$ is pertinent to the solutions of the dissimilar media problem.

First of all, since $\Phi_1(z)$ and $\Psi_1(z)$ are holomorphic in S^+ they must take the forms:

$$\Phi_1(z) = A_0 + A_1 z + A_2 z^2 + \cdots \quad \text{for } |z| < 1, \tag{13}$$

$$\Psi_1(z) = B_0 + B_1 z + B_2 z^2 + \cdots \quad \text{for } |z| < 1. \tag{14}$$

From eqns (8), (13) and (14), $\Omega_1(z)$ is found to be holomorphic in S^-. Therefore:

$$\Omega_1(z) = E_0 + \frac{E_1}{z} + \frac{E_2}{z^2} \cdots \quad \text{for } |z| > 1, \tag{15}$$

where

$$E_0 = \bar{A}_0, \quad E_1 = 0. \tag{16}$$

In the region S^-, $\Phi_2(z)$ and $\Psi_2(z)$ are holomorphic including the point at infinity, i.e.

$$\Phi_2(z) = a_0 + \frac{a_1}{z} + \frac{a_2}{z^2} + \cdots \quad \text{for } |z| > 1, \tag{17}$$

$$\Psi_2(z) = b_0 + \frac{b_1}{z} + \frac{b_2}{z^2} + \cdots \quad \text{for } |z| > 1. \tag{18}$$

Substituting eqns (17) and (18) into (8) yields:

$$\Omega_2(z) = -\frac{\bar{b}_0}{z^2} - \frac{\bar{b}_1}{z} + \xi(z) \quad \text{for } |z| < 1, \tag{19}$$

which is holomorphic in S^+ with the exception of the point $z = 0$ and $\xi(z)$ is a polynomial in positive powers of z.

Without going into details, the constants a_0 and b_0 which appeared in eqns (17) and (18), respectively can be found immediately by following the procedure described in Muskhelishivili (1953) for the case of one material. If σ_1^∞ and σ_2^∞ denote the values of the principal stresses at infinity and ω the angle made by the direction of σ_1^∞ with the x-axis, then:

$$a_0 = \tfrac{1}{4}(\sigma_1^\infty + \sigma_2^\infty) + i\frac{2\mu_2 \varepsilon^\infty}{1+\kappa_2} \tag{20}$$

$$b_0 = -\tfrac{1}{2}(\sigma_1^\infty - \sigma_2^\infty)\,\mathrm{e}^{-2i\omega} \tag{21}$$

where ε^∞ is the rotation at infinity.

In order to reduce the boundary problem to the solution of linear relationship or Hilbert problem, we must extend all complex functions $\Phi_j(z)$, $\Omega_j(z)$ into the whole region. Starting from the assumptions that the stresses and displacements are continuous over the bonded segments of the circle $|z| = 1$, eqns (11) and (12) may be regarded as the conditions of analytic continuation of $\Phi_j(z)$, $\Omega_j(z)$ from S^+ to S^- across L^*. Now, $\Phi_1(t)$, $\Omega_1(t)$ in eqns (11) and (12) may be solved explicitly in terms of $\Phi_2(t)$ and $\Omega_2(t)$, and the resulting expressions are valid everywhere in the z-plane as:

$$\Phi_1(z) = \frac{\mu_2+\mu_1\kappa_2}{\mu_2(1+\kappa_1)}\Phi_2(z) + \frac{\mu_2-\mu_1}{\mu_2(1+\kappa_1)}\Omega_2(z) + \frac{2\mu_1}{1+\kappa_1}[\beta_2 g_2'(z) - \beta_1 g_1'(z)], \quad (22)$$

$$\Omega_1(z) = \frac{\mu_2\kappa_1-\mu_1\kappa_2}{\mu_2(1+\kappa_1)}\Phi_2(z) + \frac{\mu_1+\mu_2\kappa_1}{\mu_2(1+\kappa_1)}\Omega_2(z) - \frac{2\mu_1}{1+\kappa_1}[\beta_2 g_2'(z) - \beta_1 g_1'(z)]. \quad (23)$$

Substituting eqns (13) and (19) into (22), the definition of the function $\Phi_2(z)$ can be extended into a region S^+ by allowing poles up to the second-order at $z = 0$. This gives:

$$\Phi_2(z) = \frac{T_1}{z^2} + \frac{T_2}{z} + \eta(z) \quad \text{for } |z| < 1, \quad (24)$$

where $\eta(z)$ is a function holomorphic everywhere in the region S^+, and

$$T_1 = \frac{\mu_2-\mu_1}{\mu_2+\mu_1\kappa_2}\bar{b}_0, \quad T_2 = \frac{\mu_2-\mu_1}{\mu_2+\mu_1\kappa_2}\bar{b}_1 + R_1. \quad (25)$$

Note that the constant R_1 in eqn (25) may be found from the temperature functions which possess simple poles at $z = 0$ (Chao and Shen, 1993b). Similarly, substituting eqns (15) and (17) into (23), the function $\Omega_2(z)$ can also be extended into the region $|z| > 1$ as:

$$\Omega_2(z) = R_2 z + \zeta(z), \quad (26)$$

where $\zeta(z)$ is a holomorphic function everywhere in S^- and R_2 can be determined from the temperature functions which possess simple poles at infinity (Chao and Shen, 1993b).

4. HILBERT FORMULATION

Inserting eqns (22) and (23) into the boundary conditions eqns (9) and (10) and solving them simultaneously yields:

$$[\Phi_2(t)+\Omega_2(t)]^+ + \alpha[\Phi_2(t)+\Omega_2(t)]^- = f(t), \quad (27)$$

$$[\Phi_2(t)-\alpha\Omega_2(t)]^+ - [\Phi_2(t)-\alpha\Omega_2(t)]^- = g(t), \quad (28)$$

where:

$$f(t) = \frac{\mu_2(1+\kappa_1)}{\mu_2+\mu_1\kappa_2}p^+(t) + \frac{\mu_1(1+\kappa_2)}{\mu_2+\mu_1\kappa_2}p^-(t) + \Delta[\beta_1(g_1'^+(t)-g_1'^-(t)) - \beta_2(g_2'^+(t)-g_2'^-(t))], \quad (29)$$

$$g(t) = \frac{\mu_2(1+\kappa_1)}{\mu_2+\mu_1\kappa_2}[p^+(t)-p^-(t)] + \Delta[\beta_1(g_1'^+(t)-g_1'^-(t)) - \beta_2(g_2'^+(t)-g_2'^-(t))] \quad (30)$$

$$\Delta = \frac{2\mu_1\mu_2}{\mu_2+\mu_1\kappa_2},$$

and they must satisfy the Hölder condition on L. The parameter α stands for:

$$\alpha = \frac{\mu_1+\mu_2\kappa_1}{\mu_2+\mu_1\kappa_2}.$$

Knowing that eqn (28) is a Plemelj equation for the function $\Phi_2(z)-\alpha\Omega_2(z)$, and using the property from eqn (26), we have:

$$\Phi_2(z)-\alpha\Omega_2(z) = \frac{1}{2\pi i}\int_L \frac{g(t)}{t-z}\mathrm{d}t+e_0+e_1 z+\frac{t_2}{z}+\frac{t_1}{z^2}. \tag{31}$$

Furthermore, the nonhomogeneous Hilbert eqn (27) gives:

$$\Phi_2(z)+\Omega_2(z) = \frac{X(z)}{2\pi i}\int_L \frac{f(t)}{X^+(t)(t-z)}\mathrm{d}t+X(z)\left[P_{n+1}(z)+\frac{D_1}{z}+\frac{D_2}{z^2}\right], \tag{32}$$

where the Plemelj function:

$$X(z) = \prod_{m=1}^{n}(z-a_m)^{-1/2+i\beta}(z-b_m)^{-1/2-i\beta}, \quad m = 1,2,\ldots,n. \tag{33}$$

and the polynomial $P_{n+1}(z)$ is of degree not greater than $n+1$, i.e.

$$P_{n+1}(z) = c_0+c_1 z+\cdots+c_n z^n+c_{n+1}z^{n+1}. \tag{34}$$

The exponent β which appeared in eqn (33) is:

$$\beta = \frac{1}{2\pi}\log\alpha$$

which is referred to as a bi-elastic constant. It is realized that the singularities of the thermal stresses are the same as those for the isothermal problem which would not be affected by the discontinuous jumps of the thermal properties across the interface.

By means of eqns (22), (23), (31) and (32) the general solution involving the four unknown functions $\Phi_j(z)$, $\Omega_j(z)$, $(j = 1, 2)$ may be arranged into a compact form as follows:

$$\Phi_1(z) = \frac{(\mu_2+\mu_1\kappa_2)[\mu_2(1+\kappa_1)F_1(z)+\mu_1(1+\kappa_2)F_2(z)]}{\mu_2(1+\kappa_1)[\mu_1(1+\kappa_2)+\mu_2(1+\kappa_1)]}+\frac{2\mu_1}{1+\kappa_1}[\beta_2 g_2'(z)-\beta_1 g_1'(z)], \tag{35}$$

$$\Omega_1(z) = \frac{\mu_2(1+\kappa_1)(\mu_1+\mu_2\kappa_1)F_1(z)-\mu_1(1+\kappa_2)(\mu_2+\mu_1\kappa_2)F_2(z)}{\mu_2(1+\kappa_1)[\mu_1(1+\kappa_2)+\mu_2(1+\kappa_1)]} -\frac{2\mu_1}{1+\kappa_1}[\beta_2 g_2'(z)-\beta_1 g_1'(z)], \tag{36}$$

and

$$\Phi_2(z) = \frac{(\mu_1+\mu_2\kappa_1)F_1(z)+(\mu_2+\mu_1\kappa_2)F_2(z)}{\mu_1(1+\kappa_2)+\mu_2(1+\kappa_1)} \tag{37}$$

$$\Omega_2(z) = \frac{(\mu_2+\mu_1\kappa_2)[F_1(z)-F_2(z)]}{\mu_1(1+\kappa_2)+\mu_2(1+\kappa_1)}, \tag{38}$$

where

$$F_1(z) = \frac{X(z)}{2\pi i}\int_L \frac{f(t)}{X^+(t)(t-z)}\mathrm{d}t + X(z)\left[P_{n+1}(z) + \frac{D_1}{z} + \frac{D_2}{z^2}\right], \tag{39}$$

$$F_2(z) = \frac{1}{2\pi i}\int_L \frac{g(t)}{t-z}\mathrm{d}t + e_0 + e_1 z + \frac{t_2}{z} + \frac{t_1}{z^2}. \tag{40}$$

The quantities t_1 and t_2 in eqn (31) may be associated with the coefficients of the series given by eqns (19) and (24) as:

$$t_1 = T_1 + \alpha\bar{b}_0, \quad t_2 = T_2 + \alpha\bar{b}_1. \tag{41}$$

Using eqns (17) and (26), the constants e_1 and c_{n+1} appearing in eqns (31) and (32), respectively can be expressed as:

$$e_1 = -\alpha R_2, \tag{42}$$

$$c_{n+1} = \frac{R_2}{\lim\limits_{z\to\infty}[z^n X(z)]}. \tag{43}$$

The problem is now reduced to the determination of $n+5$ unknown constants e_0, b_1, D_1, D_2, c_m $(m = 0, n)$ appearing in eqns (39) and (40) which must be solved by the additional $n+5$ equations to determine the remaining unknown constants uniquely. Using the behavior of the stress function $\Phi_2(z)+\Omega_2(z)$ near the point $|z| = 0$ in conjunction with eqns (19) and (24), there follows:

$$X(z)\left(\frac{D_1}{z} + \frac{D_2}{z^2}\right) = \frac{T_1 - \bar{b}_0}{z^2} + \frac{T_2 - \bar{b}_1}{z}. \tag{44}$$

This gives two equations for solving the unknown constants D_1 and D_2 in terms of b_1. In addition, the constants e_0, c_n and c_{n-1} can also be expressed in terms of b_1 by applying the behavior of the stress functions $\Phi_2(z)$, $\Omega_2(z)$, for large values of $|z|$ in conjunction with eqn (16). Now, the rest of the n unknown constants $b_1, c_0, c_1, \ldots, c_{n-2}$ are to be found from the conditions that the displacements must be single-valued. Applying eqns (7) and (8), such a requirement is equivalent to:

$$\frac{1}{\mu_1}\left[\int_{L_m}\kappa_1\Phi_1^+(t)\,\mathrm{d}t - \int_{L_m}\Omega_1^-(t)\,\mathrm{d}t\right] - \frac{1}{\mu_2}\left[\int_{L_m}\kappa_2\Phi_2^-(t)\,\mathrm{d}t - \int_{L_m}\Omega_2^+(t)\,\mathrm{d}t\right]$$
$$+2\beta_1\int_{L_m} g_1'^+(t)\,\mathrm{d}t - 2\beta_2\int_{L_m} g_2'^-(t)\,\mathrm{d}t = 0, \quad m = 1,2,\ldots,n. \tag{45}$$

For the purpose of computation in subsequent work, eqns (22) and (23) may be used to put eqn (45) in the form:

$$\int_{L_m}\{\kappa_1(\mu_2+\mu_1\kappa_2)[\Phi_2^+(t)-\Phi_2^-(t)] + (\mu_1+\mu_2\kappa_1)[\Omega_2^+(t)-\Omega_2^-(t)]$$
$$+2\mu_1\mu_2[\kappa_1\beta_2(g_2'^+(t)-g_2'^-(t)) + \beta_1(g_1'^+(t)-g_1'^-(t))]\}\,\mathrm{d}t = 0, \quad m = 1,2,\ldots,n. \tag{46}$$

This gives a system of n linear equations solving for the n unknown coefficients b_1, c_0, $c_1, \ldots, c_{n-2}$. Now, we have completed the general solution to the given thermoelastic problem once the singular integrals with kernels of Cauchy type as indicated in eqns (31) and (32) are evaluated. Note that the thermoelastic field presented above may become invalid for certain combinations of the angle of heat flux and the central angle subtended by the circular-arc crack. The condition of fully open crack displacement must be addressed to validate the given results.

5. SINGLE CIRCULAR-ARC CRACK

For illustrating the use of the present approach, we now consider the problem of an infinite plate with a circular-arc crack lying along the interface of a unit disk as shown in Fig. 2. The applied loads at infinity consist of uniform tension, p, directed at an angle ω and uniform heat flux, q_0, directed at an angle γ with respect to the x-axis. For an insulated crack, the temperature functions are given by (Chao and Shen, 1993b)

$$g_1'(z) = \frac{1}{2}\left[w_1 z - \frac{w_2}{z} + \frac{k_2}{k_1}\left(w_3\sqrt{z^2-2\cos\theta z+1} - \frac{w_4\sqrt{z^2-2\cos\theta z+1}}{z}\right)\right], \quad (47)$$

$$g_2'(z) = \frac{1}{2}\left[w_1 z - \frac{w_2}{z} + \left(w_3\sqrt{z^2-2\cos\theta z+1} - \frac{w_4\sqrt{z^2-2\cos\theta z+1}}{z}\right)\right] \quad (48)$$

where

$$w_1 = \frac{2k_2}{k_1+k_2}\Gamma_0,$$

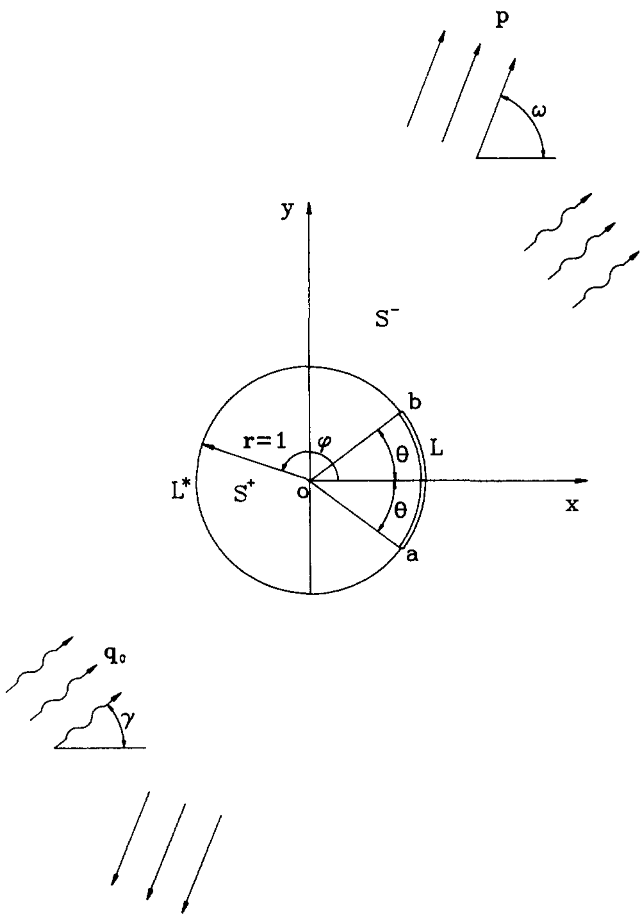

Fig. 2. Single circular-arc crack under uniform tension and heat flux.

$$w_2 = \frac{-2k_2}{k_1+k_2}\Gamma_0,$$

$$w_3 = \frac{2k_1}{k_1+k_2}\Gamma_0,$$

$$w_4 = \frac{-2k_1}{k_1+k_2}\Gamma_0,$$

$$\Gamma_0 = -\frac{q_0 e^{-i\gamma}}{k_2}.$$

Accordingly, the constants R_1 in eqn (25) and R_2 in eqn (26) become:

$$R_1 = \frac{\mu_1\mu_2}{\mu_2+\mu_1\kappa_2}\beta_2(w_2-w_4), \tag{49}$$

$$R_2 = \frac{\mu_1\mu_2}{\mu_1+\mu_2\kappa_1}\left[\beta_2(w_1+w_3)-\beta_1\left(\frac{k_2}{k_1}w_3+w_1\right)\right]. \tag{50}$$

Since the ends of the crack, L, are located at $a = \exp(-i\theta)$ and $b = \exp(i\theta)$ on $|z| = 1$, the Plemelj function in eqn (33) yields:

$$X(z) = (z-e^{i\theta})^{-1/2-i\beta}(z-e^{-i\theta})^{-1/2+i\beta}. \tag{51}$$

For the traction free condition, $p^+(t) = p^-(t) = 0$, the line integrals that appeared in eqns (39) and (40) can be evaluated by residual theory and the results are:

$$\frac{1}{2\pi i}\int_L \frac{g(t)}{t-z}\mathrm{d}t = \frac{\Delta\left(\beta_1\frac{k_2}{k_1}-\beta_2\right)}{2}\left[w_3\sqrt{z^2-2\cos\theta z+1}\right.$$
$$\left.-w_3(z-\cos\theta)-\omega_4\frac{\sqrt{z^2-2\cos\theta z+1}}{z}-\frac{w_4}{z}+w_4\right], \tag{52}$$

$$\frac{X(z)}{2\pi i}\int_L \frac{f(t)}{X^+(t)(t-z)}\mathrm{d}t = \frac{\Delta\left(\beta_1\frac{k_2}{k_1}-\beta_2\right)}{1+\alpha}\left\{w_3\left[\sqrt{z^2-2\cos\theta z+1}\right.\right.$$
$$\left.-\left(z^2-2(\cos\theta-\beta\sin\theta)z+\frac{2\beta^2+1}{2}(1-\cos 2\theta)\right)X(z)\right]$$
$$\left.-w_4\left[\frac{\sqrt{z^2-2\cos\theta z+1}}{z}+\left(\frac{1}{zX(0)}-z+2(\cos\theta-\beta\sin\theta)\right)X(z)\right]\right\}, \tag{53}$$

where

$$\Delta = \frac{2\mu_1\mu_2}{\mu_2+\mu_1\kappa_2}, \quad \alpha = \frac{\mu_1+\mu_2\kappa_1}{\mu_2+\mu_1\kappa_2}.$$

Expanding eqn (51) near $z = 0$ renders:

$$X(z) = -e^{2\beta\theta}[1+M_1 z+M_2 z^2+\cdots],$$

where

$$M_1 = \cos\theta+2\beta\sin\theta, \quad M_2 = \frac{3\cos 2\theta+1}{4}+\beta^2(1-\cos\theta)+2\beta\sin 2\theta.$$

The constants D_1 and D_2 can be obtained in a straightforward manner from eqn (44). The results are:

$$D_2 = e^{-2\beta\theta}\frac{\mu_1(1+\kappa_2)}{\mu_2+\mu_1\kappa_2}\bar{b}_0, \tag{54}$$

$$D_1 = e^{-2\beta\theta}\left[\frac{\mu_1(1+\kappa_2)}{\mu_2+\mu_1\kappa_2}(\bar{b}_1-M_1\bar{b}_0)+R_1\right] \tag{55}$$

where:

$$b_0 = -\frac{p}{2}e^{-2i\omega},$$

$$b_1 = -\kappa_2\bar{a}_1-\bar{H} = -\kappa_2\bar{F},$$

$$H = \mu_2\beta_2\left[-w_2+\frac{k_2}{k_1}\left(w_3\frac{\sin^2\theta}{2}+w_4\cos\theta\right)\right].$$

Note that the unknown constant F is equivalent to the resultant force exerted on L due to the thermal load which will be determined later. Expanding eqn (51) for large $|z|$:

$$X(z) = \frac{1}{z}+\frac{N_1}{z^2}+\frac{N_2}{z^3}+\cdots$$

where

$$N_1 = \cos\theta-2\beta\sin\theta, \quad N_2 = \tfrac{1}{4}+\beta^2+(\tfrac{3}{4}-\beta^2)\cos 2\theta-2\beta\sin 2\theta.$$

The constant c_2 $(n = 1)$ which appeared in eqn (43) becomes:

$$c_2 = R_2. \tag{56}$$

Furthermore, as $|z| \to \infty$:

$$\Phi_2(z) \to \frac{p}{4}+\frac{F-H/\kappa_2}{z}+O\left(\frac{1}{z^2}\right). \tag{57}$$

Putting eqn (37) into eqn (57) and using the property with the knowledge that for large $|z|$:

$$\frac{1}{2\pi i}\int_L \frac{g(t)}{t-z}\mathrm{d}t \to \mathrm{O}\left(\frac{1}{z}\right),\quad \frac{X(z)}{2\pi i}\int_L \frac{f(t)}{X^+(t)(t-z)}\mathrm{d}t \to \mathrm{O}\left(\frac{1}{z^2}\right)$$

yields the following equations:

$$R_3+c_0+N_1c_1 = \frac{(1+\kappa_2)(\mu_1+\mu_2+\mu_2\kappa_1)F-[\mu_1(1+\kappa_2)+\mu_2(1+\kappa_1)]H/\kappa_2}{\mu_1+\mu_2\kappa_1}, \tag{58}$$

$$\frac{(\mu_2+\mu_1\kappa_2)e_0}{\mu_1(1+\kappa_2)+\mu_2(1+\kappa_1)}+\frac{(\mu_1+\mu_2\kappa_1)(c_1+N_1R_2)}{\mu_1(1+\kappa_2)+\mu_2(1+\kappa_1)} = \frac{p}{4}, \tag{59}$$

where

$$R_3 = \frac{(\mu_2+\mu_1\kappa_2)}{\mu_1+\mu_2\kappa_1}\frac{\Delta\left(\frac{k_2}{k_1}\beta_1-\beta_2\right)}{2}\left[\frac{1-\cos 2\theta}{4}w_3-w_4(1-\cos\theta)\right]+N_2R_2. \tag{60}$$

Now, the supplementary condition equivalent to eqns (13), (15) and (16) gives:

$$\Phi_1(0) = \bar{\Omega}_1(\infty). \tag{61}$$

Inserting eqns (35) and (36) into eqn (61) yields the following equation:

$$(\mu_2+\mu_1\kappa_2)\left[\frac{\mu_1(1+\kappa_2)}{\mu_2(1+\kappa_1)}\bar{e}_0-\mathrm{e}^{2\beta\theta}\bar{c}_0\right]-\mu_1(1+\kappa_2)[-M_1\kappa_2\bar{\bar{F}}$$
$$+(M_2-M_1^2)b_0]+R_4 = (\mu_1+\mu_2\kappa_1)c_1-(\mu_2+\mu_1\kappa_2)\left[\frac{\mu_1(1+\kappa_2)}{\mu_2(1+\kappa_1)}\right]e_0 \tag{62}$$

where:

$$R_4 = (\mu_2+\mu_1\kappa_2)\left\{M_1R_1+\frac{\Delta\left(\frac{k_2}{k_1}\beta_1-\beta_2\right)}{1+\alpha}\left\{w_3\left[-1+\mathrm{e}^{2\beta\theta}(2\beta^2+1)\frac{(1-\cos\theta)}{2}\right]\right.\right.$$
$$\left.\left.+2\,\mathrm{e}^{2\beta\theta}w_4(\cos\theta-\beta\sin\theta)\right\}\right\}+(\mu_2+\mu_1\kappa_2)\frac{\mu_1(1+\kappa_2)}{\mu_2(1+\kappa_1)}\frac{\Delta\left(\frac{k_2}{k_1}\beta_1-\beta_2\right)}{2}$$
$$\times(1-\cos\theta)(w_4-w_3)+M_1R_2. \tag{63}$$

Now, we have three equations, (58), (59) and (62), for solving four unknowns e_0, c_0, c_1 and F. The remaining equation can be found from the condition that displacement must be single-valued. Applying eqn (46), we have:

$$\frac{(1+\kappa_1)(\mu_2+\mu_1\kappa_2)(\mu_1+\mu_2\kappa_1)}{\mu_1(1+\kappa_2)+\mu_2(1+\kappa_1)}(2c_0+2N_1c_1+2\,\mathrm{e}^{2\beta\theta}D_1+2M_1\,\mathrm{e}^{2\beta\theta}D_2) = T_e+R_5. \tag{64}$$

where:

$$T_e = 2\mu_1\mu_2\left(\kappa_1\beta_2+\beta_1\frac{k_2}{k_1}\right)\left[(1-\cos\theta)w_4-\frac{(1-\cos 2\theta)}{4}w_3\right], \tag{65}$$

$$R_5 = \frac{(\mu_2+\mu_1\kappa_2)(\mu_1+\mu_2\kappa_1)}{\mu_1(1+\kappa_2)+\mu_2(1+\kappa_1)}\left\{\Delta\left(\frac{k_2}{k_1}\beta_1-\beta_2\right)\left(\frac{\kappa_1(\mu_2+\mu_1\kappa_2)}{2(\mu_1+\mu_2\kappa_1)}-\frac{1}{2}\right)\right.$$
$$\left.\left[2(1-\cos\theta)w_4-\frac{1-\cos 2\theta}{2}w_3\right]-2(1+\kappa_1)M_2R_2\right\}. \tag{66}$$

After some algebraic manipulations, the final explicit forms are obtained as:

$$F = \frac{[\mu_1(1+\kappa_2)+\mu_2(1+\kappa_1)][T_e+R_5+(1+\kappa_1)(\mu_2+\mu_1\kappa_2)H/\kappa_2]}{2(1+\kappa_1)(1+\kappa_2)[(\mu_2+\mu_1\kappa_2)(\mu_1+\mu_2+\mu_2\kappa_1)-\mu_1\kappa_2(\mu_1+\mu_2\kappa_1)]}$$
$$+\frac{(1+\kappa_1)(\mu_1+\mu_2\kappa_1)(\mu_2+\mu_1\kappa_2)(2R_3-2R_1)}{2(1+\kappa_1)(1+\kappa_2)[(\mu_2+\mu_1\kappa_2)(\mu_1+\mu_2+\mu_2\kappa_1)-\mu_1\kappa_2(\mu_1+\mu_2\kappa_1)]}. \tag{67}$$

$$c_1 = \frac{\mu_2(1+\kappa_1)}{-N_1\mu_2(1+\kappa_1)(\mu_2+\mu_1\kappa_2)\,e^{2\beta\theta}+(\mu_1+\mu_2\kappa_1)[\mu_2(1+\kappa_1)+2\mu_1(1+\kappa_2)]}$$
$$\times\left\{\mu_1(1+\kappa_2)(M_1^2-M_2)\mathscr{R}[b_0]+\frac{\mu_1(1+\kappa_2)}{\mu_2(1+\kappa_1)}\left\{\frac{p}{2}[\mu_1(1+\kappa_2)+\mu_2(1+\kappa_1)]\right.\right.$$
$$\left.-2N_1(\mu_1+\mu_2\kappa_1)\mathscr{R}[R_2]\right\}+\mathscr{R}[R_4]+e^{2\beta\theta}(\mu_2+\mu_1\kappa_2)\mathscr{R}[R_3]$$
$$+(1+\kappa_2)\left[M_1\mu_1\kappa_2-\frac{e^{2\beta\theta}(\mu_2+\mu_1\kappa_2)(\mu_1+\mu_2+\mu_2\kappa_1)}{(\mu_1+\mu_2\kappa_1)}\right]\mathscr{R}[F]$$
$$\left.+\frac{\mu_1(1+\kappa_2)+\mu_2(1+\kappa_1)}{\kappa_2(\mu_1+\mu_2\kappa_1)}\mathscr{R}[H]\right\}+\frac{i}{(\mu_1+\mu_2\kappa_1)+N_1(\mu_2+\mu_1\kappa_2)\,e^{2\beta\theta}}$$
$$\times\left\{\mu_1(1+\kappa_2)(M_1^2-M_2)\mathscr{T}[b_0]+\mathscr{T}[R_4]-e^{2\beta\theta}(\mu_2+\mu_1\kappa_2)\mathscr{T}[R_3]-(1+\kappa_2)\right.$$
$$\left.\times\left[M_1\mu_1\kappa_2-\frac{e^{2\beta\theta}(\mu_2+\mu_1\kappa_2)(\mu_1+\mu_2+\mu_2\kappa_1)}{(\mu_1+\mu_2\kappa_1)}\right]\mathscr{T}[F]-\frac{\mu_1(1+\kappa_2)+\mu_2(1+\kappa_1)}{\kappa_2(\mu_1+\mu_2\kappa_1)}\mathscr{T}[H]\right\}. \tag{68}$$

$$c_0 = -N_1c_1+\frac{(1+\kappa_2)(\mu_1+\mu_2+\mu_2\kappa_1)F-[\mu_1(1+\kappa_2)+\mu_2(1+\kappa_1)]H/\kappa_2}{\mu_1+\mu_2\kappa_1}-R_3 \tag{69}$$

$$e_0 = \frac{p}{4}\left[\frac{\mu_1(1+\kappa_2)+\mu_2(1+\kappa_1)}{\mu_2+\mu_1\kappa_2}\right]-\frac{\mu_1+\mu_2\kappa_1}{\mu_2+\mu_1\kappa_2}(c_1+N_1R_2), \tag{70}$$

where $\mathscr{R}$ and $\mathscr{T}$ denote the real and imaginary part of the complex functions, respectively.

Up to now, all the coefficients that appeared in eqns (31) and (32) have been determined. From eqn (51), it can be found that the stresses and displacements oscillate violently as the crack tip is approached and this indicates that the upper and lower faces of the crack will overlap one another. However, this unrealistic aspect of the solution is confined to very small regions near the crack ends. Based on this finding, we would not further discuss the fully open crack assumption near the crack tips. Although the overlap of the displacements near the crack tips may be permissible from the mathematical formulation of the problem, it is still required that the radial component of relative displacements of the two faces of the crack must be greater than zero. The derivative of the relative radial displacement of the two faces of the crack can be expressed as:

$$u_r'(t) = \mathscr{R}\{[u'(t)+iv'(t)]/t\}, \tag{71}$$

where

$$u'(t)+iv'(t) = \frac{1}{\mu_2}[\kappa_2\Phi_2^-(t)-\Omega_1^+(t)+2\mu_2\beta_2 g_2'^-(t)] - \frac{1}{\mu_1}[\kappa_1\Phi_1^+(t)-\Omega_1^-(t)+2\mu_1\beta_1 g_1'^+(t)]$$
$$= \frac{(\mu_1+\mu_2\kappa_1)(1+\kappa_1)(\mu_2+\mu_1\kappa_2)[F_1^-(t)-F_1^+(t)]}{\mu_1\mu_2(1+\kappa_1)[\mu_1(1+\kappa_2)+\mu_2(1+\kappa_1)]}$$
$$+\frac{\mu_1(\mu_2+\mu_1\kappa_2)(\kappa_1\kappa_2-1)[F_2^-(t)-F_2^+(t)]}{\mu_1\mu_2(1+\kappa_1)[\mu_1(1+\kappa_2)+\mu_2(1+\kappa_1)]}. \tag{72}$$

The condition of the fully open crack displacement is stated as:

$$\int_{e^{-i\theta}}^{t} u_r'(t')\,dt' \geqslant 0 \quad \text{for } t \in L. \tag{73}$$

Now, the above results satisfying eqn (73) constitute a closed form solution to the given problem which validates the assumption of the fully open crack. When the crack surfaces come into contact with one another over a certain length, a much more complicated boundary value problem than the present one must be considered. We do not further enter into the subject in this paper.

According to the concept of fracture mechanics, K_1 and K_2 may be considered as the stress intensity factors that cause unstable crack extension upon reaching some critical values. In the usual manner, they can be obtained as:

$$K_1 - iK_2 = 2\sqrt{2\pi}\,e^{\beta\pi}\lim_{z\to z_1}(z-z_1)^{1/2+i\beta}\Phi_1(z). \tag{74}$$

In order to treat the problem by making use of eqn (74), the coordinate must be rotated such that the crack tip is parallel to the x-axis. The convenient transformation for this purpose is:

$$z = i e^{i\theta}(z'-i-\sin\theta\cos\theta), \tag{75}$$

at point b. Substituting eqns (75) and (35) into eqn (74), the stress intensity factors at point b are:

$$K_1 - iK_2 = \frac{-2\sqrt{\pi}\exp\left[\beta(\theta+\pi)+i\left(\frac{\pi}{2}-\frac{\theta}{2}+\beta\log(2\sin\theta)\right)\right]}{(1+\alpha)\sqrt{\sin\theta}}$$
$$\times\left\{\frac{\Delta\left(\beta_1\frac{k_2}{k_1}-\beta_2\right)}{1+\alpha}\left[-w_3\left(e^{2i\theta}-2(\cos\theta-\beta\sin\theta)\,e^{i\theta}+\frac{1+2\beta^2}{2}(1-\cos 2\theta)\right)\right.\right.$$
$$\left.\left.+w_4(e^{-2\beta\theta-i\theta}+e^{i\theta}-2(\cos\theta-\beta\sin\theta))\right]+c_0+c_1 e^{i\theta}+c_2 e^{2i\theta}+D_1 e^{-i\theta}+D_2 e^{-2i\theta}\right\}. \tag{76}$$

It is seen that the stress intensity factors are dependent on the angle of heat flow and heat conductivity as well as elastic and thermal constants. In the special case for isothermal elasticity, eqn (76) reduces to the results given by Perlman and Sih (1967a).

Table 1. Typical properties of materials for composites

Properties	Inclusion			Matrix
	Boron	Carbon	Glass	Epoxy
Shear modulus, GN m^{-2}	172.5	138.7	31.74	1.24
Poisson's ratio	0.2	0.2	0.2	0.4
Thermal expansion coefficient, $10^{-6}\,°C^{-1}$	5.0	2.7	5.0	57.6
Heat conductivity, W (m °C)$^{-1}$	18.2	15.6	1.94	0.45

6. NUMERICAL EXAMPLES AND DISCUSSION

In the following work, three typical examples of composite materials are given to illustrate interface tractions along the bonded region as well as the stress intensity factors. Let the material of the surrounding matrix be epoxy and the inclusion be glass, carbon or boron, respectively. All the thermoelastic properties of composite materials are listed in Table 1.

Applying eqn (73), the critical values of the crack angle θ for which the crack faces come into contact at a certain point can be determined for different combinations of materials and the angle of heat flux under plane strain conditions. The results are plotted

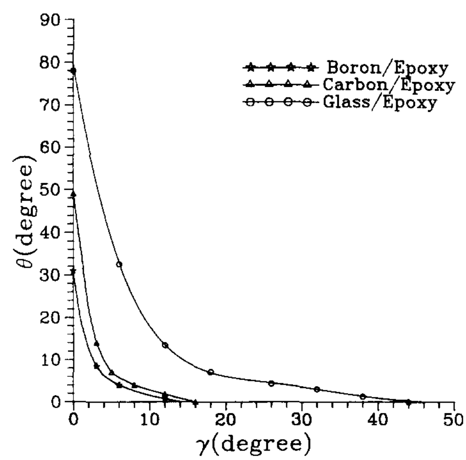

Fig. 3. Variation of the critical value of θ with γ, for which the crack faces come into contact at a certain point.

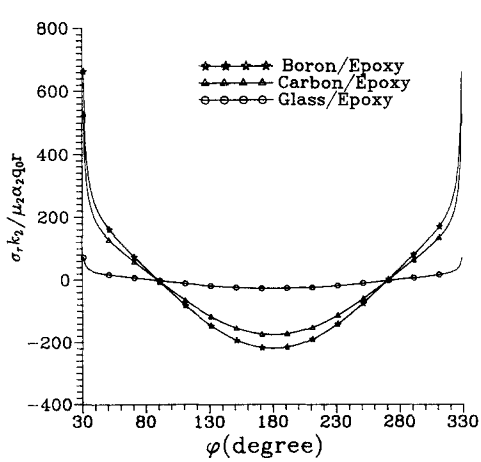

Fig. 4. The nondimensional radial stress σ_r in bond, $\theta = 30°$, $\gamma = 0°$.

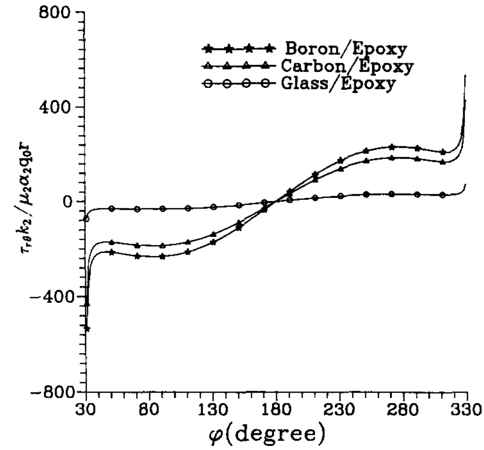

Fig. 5. The nondimensional shear stress $\tau_{r\theta}$ in bond, $\theta = 30°$, $\gamma = 0°$.

in Fig. 3 which reveals that the solutions derived above are valid only for those values of θ and γ in the region enclosed by each curve and the γ-axis. It is shown that, when the angle of heat flux $\gamma = 0°$, the critical value of the crack angle θ is equal to 78° for glass/epoxy composite while $\theta = 49°$ and 31° for carbon/epoxy and boron/epoxy composites, respectively. As the angle of heat flux shifts from 0° to 50° or above, the relative radial displacement u_r takes a negative value on a certain part of the boundary for any value of the crack angle and this implies that there is an overlap of opposite faces of the crack. Figures 4 and 5 indicate the nondimensional tractions along the bonded portion for different composite materials with the crack angle $\theta = 30°$ under remote heat flux approached from the negative x-axis. The results show that the magnitude of interface tractions increases with the rigidity of the inclusion. Namely, interface tractions on the bonded region will be enhanced when the fiber is made more rigid than the matrix. It should be noted that the positively singular traction σ_r always prevails for each composite material which validates the assumption of fully open crack. As the definition from eqn (74), the stress intensity factors are introduced to measure the local energy intensification in the vicinity of the crack tip. Both the nondimensional stress intensity factors K_1 and K_2 vs the crack angle θ are plotted in Figs 6 and 7, respectively for the angle of heat flux $\gamma = 0°$. Due to symmetric property, only one of the crack tips needs to display the factors K_1 and K_2. It is found that both K_1 and K_2 increase with the rigidity of the fiber for the given crack dimension and load angle. It is interesting

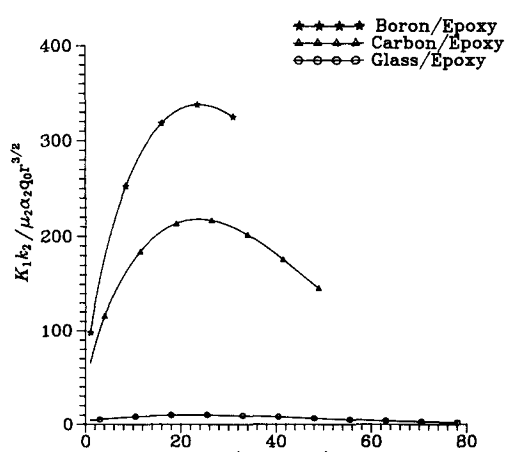

Fig. 6. Variation of the nondimensional stress intensity factor K_1 with the crack angle for $\gamma = 0°$.

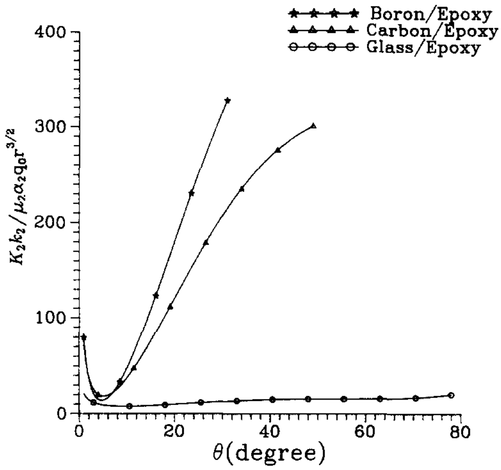

Fig. 7. Variation of the nondimensional stress intensity factor K_2 with crack angle for $\gamma = 0°$.

to note that the stress intensity factors would not monotonically increase with the crack angle θ. Referring to Fig. 6, the factor K_1 for boron/epoxy composite begins to increase with the crack angle θ and attains its maximum value around $\theta = 24°$ and then decreases as the crack angle further increases up to $\theta = 31°$. From a failure analysis point of view, system instability for boron/epoxy composite under remote heat flux ($\gamma = 0°$) would be likely to take place as the interface crack extends along the curved bond up to 24° where the resulting stress intensity factor is largest. As the crack extends much further and beyond 32° a negatively singular traction may prevail and crack arrest might occur. Similar observations can also be applied for carbon/epoxy and glass/epoxy composites except the crack angle beyond 49° and 78° for carbon/epoxy and glass/epoxy composites, respectively, where the initial assumption of the fully open crack is no longer valid.

7. CONCLUSION

A general solution is obtained to the thermoelastic problem of curvilinear cracks in bonded dissimilar materials. The analysis was based upon the Hilbert problem formulation and a special technique of analytical continuation. In order to illustrate the use of the present approach, detailed results are given for a single circular-arc crack lying along the interface between dissimilar materials. Explicit forms of the stress functions in both the inclusion and the surrounding matrix have been provided. Comparison with the solution found for the special case shows that the solution presented here is exact and general. Three typical examples of composite materials are given to illustrate interface tractions along the bonded region as well as the stress intensity factors under remote heat flux. The condition of the fully open crack is also addressed in this study which validates the above results.

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